International Journal of Theoretical Physics, Vol. 27, No. 10, 1988

The Reality and Dimension of Space and the Complexity of Quantum Mechanics

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Received June 30, 1987

The dimension (and signature) of space is a result of distances being real numbers and quantum mechanical state functions being complex ones; it is an inescapable consequence of quantum mechanics and group theory. So nonrelativistic quantum mechanics cannot be complete (it requires *ad hoc* additional assumptions) and consistent (nor can classical physics), leading to relativity, quantum mechanics, and field theory. Implications of the constraints of consistency and physical reasonableness and of group theory for the structure of these theories are considered. It appears that there are simple, perhaps unavoidable reasons for the laws of physics, the nature of the world they describe, and the space in which they act.

1. INTRODUCTION

Physics, ultimately, is based on experiment. However, what experiment has revealed is that the physical universe is governed by simple, coherent laws—suggesting that coherence places strong restraints on the laws. Here we consider how some of the most basic properties of physical theory, which seem to follow inescapably from experiment, determine fundamental properties of space—the dimension and signature—and imply strong constraints on how we describe nature.

In the next section the dimension and signature of space (Mirman, 1984a,b, 1986, 1988a,b) are found, using simple group-theoretic results. They are not only unique, but in a sense improbable. There are several conditions, each requiring integer solutions. There seems no reason why any should have such solutions. Yet all do, and the values are the same. This has implications that one can only wonder at but certainly not explain.

Higher dimensional spaces are not possible. Yet there are theories using them. Is there a contradiction? Consistent theories giving a partial description of a subject can be constructed. However, consistent partial

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0020-7748/88/1000-1257$06.00/0

descriptions do not imply that consistent complete theories are possible. Theories for higher dimensional spaces may be consistent descriptions of what they have treated. But consistent complete theories are not possible in higher dimensional spaces. Incomplete theories can be consistent, but not completed.

This does not imply that the work using higher dimensional theories is incorrect. That the inconsistencies were not previously discovered means only that the aspect of the theories in which they appear have not yet been considered.

The view of physics presented here is highly conservative; the fundamental properties of space and matter that almost everyone would accept as true and necessary are so—the results follow from the most basic principles of quantum mechanics. I use the invariance of physical laws (such as the Dirac equation) under rotations and the need for quantities—mass, length, angle, and so on—that are invariant under these transformations. The results obtained thus hold for the standard theories of physics (such as the Dirac equation), although they are so general that it is difficult to imagine theories (that are in agreement with the broadest experimental results, such as invariance) for which they would not hold—such theories may not be possible. It may be that only with the conditions everyone assumes is physics possible.

That the signature of space is determined (the dimension is $3+1$, not 4) means that quantum mechanics requires relativistic quantum mechanics. In Section 3 I discuss why. But this raises the question of whether classical physics is possible in a space of different dimensionality, and indeed whether it is possible at all. I do not think so and explain (the meaning of my negative view) in Section 4.

If quantum mechanics and relativistic quantum mechanics (and field theory) are all required by consistency and coherence, then (some of) their properties should also be. How can we find these? The arguments, which require detailed analyses, are only outlined. But it could be, as discussed in Section 5, that all their properties may follow from the most fundamental requirements, such as consistency. The postulates of these theories are experimental, but perhaps not only that. This is worth more discussion than it has received. A reason for this section is to stimulate that discussion.

The assumptions on which the results are based are stated explicitly; but, as in any physical theory, the most fundamental ones are usually those not realized. It is important to try to make these explicit (so making the assumptions on which quantum mechanics is based explicit). To stimulate this analysis, we indicate in Section 6 what some of these may be. Of course the extensive discussion and consideration that this requires is not possible here; the purpose of this section is not to give all assumptions, but rather to indicate some of the topics needing further study. Undoubtedly the most important of these topics are still hidden; it is unlikely that anyone would claim to understand the real reasons for quantum mechanics.

In Section 7 results and implications are summarized.

2. THE DIMENSION

Quantum mechanics describes any object by a complex function over space, the state function; the coordinates on which it depends are real. If we transform (rotate) the coordinates, the state function is also transformed (its components are mixed); a function describing orbital angular momentum along the z axis goes into a different function in the new coordinate system; otherwise it would (incorrectly) give the angular momentum along the z' axis.

Suppose that space were n-dimensional (with any signature). For any rotation in the xy plane there is a transformation, one for each (small) angle, of the state function. For rotations in the yz plane there is another, different, set of transformations, and so on. Further, the state-function transformation corresponding to the product of two rotations should equal the product of the two state-function transformations that go with the two rotations. So the algebras of the groups transforming the coordinates and the state functions must be isomorphic (the groups are then homomorphic).

The state functions are complex, and so are transformed by a unitary group (the factor group obtained by dividing out any subgroup independent of rotations). For reasons discussed below, they cannot be transformed only by a subgroup of a unitary group. The coordinates on which the state function depends are real, and so are transformed by an orthogonal (rotation) group.

Thus, in this n-dimensional space there must be a unitary group whose algebra is isomorphic to the algebra of the n-dimensional orthogonal group $O(n)$. This is possible only for one value of n, so it is the only one that allows quantum mechanics.

A necessary, but not sufficient, condition for two algebras to be isomorphic is that the numbers of commuting generators, the ranks (denoted by ν's), and the orders, the numbers of generators, be equal. We briefly review the formulas for the orders.

Group $O(n)$ consists of products of rotations in perpendicular planes. Each axis lies in $n-1$ planes. There are n axes, giving $n(n-1)$ planes, each counted twice. The number of planes, and generators, is $n(n-1)/2$. Rotations in intersecting planes (xy and yz) do not commute, those in nonintersecting planes (xy and wz) do. Every plane in which the y axis lies intersects the xy plane. However, z, provided there are additional dimensions, lies in a plane not intersecting xy. For the number of commuting operators count every other axis, giving $n/2$ for n even and $(n-1)/2$ for n odd. Addition of two dimensions adds one more plane. So $n=2\nu$ or $n=2\nu+1$ and the number of generators is $\nu(2\nu-1)$ for n even and $\nu(2\nu+1)$ for n odd.

The unitary group on a state function of n components has $n-1$ transformations varying the $n-1$ phases (and one for the overall phase) plus $n(n-1)/2$ rotations. The relative phase of the two components affected by a rotation can be changed after the rotation by another rotation, giving an additional $n(n-1)/2$ transformations, for a total of n^2. But the one changing the overall phase commutes with the others. Discarding it gives n^2-1 generators of the unimodular unitary group in n dimensions. How many commute? The phase changes do. Rotations in intersecting planes and a phase change and rotation involving the same variable do not. The largest commuting set is that of the phase changes ($n-1$ are independent). There are $n-1$ $(=\nu)$ commuting generators.

So the orders of the unitary and orthogonal algebras are $(\nu+1)^2-1$ and $\nu(2\nu+1)$, respectively.

Thus, $(\nu+1)^2-1=\nu(2\nu\pm 1)$, giving $\nu=1$ (for B), and 3 (for D). For $\nu=1$, $SU(2)$ and $O(3)$ and also $SU(1,1)$ and $O(2,1)$ are homomorphic (their algebras are isomorphic). For $\nu=3$ there is a homomorphism between $SU(4)$ and $O(6)$. Besides groups over real numbers, there are ones over complex numbers. The algebra of $O(4)$ is not simple, but the complex extension is the algebra of the orthogonal group (the Lorentz group); complex state functions transform under $SL(2, C)$. So $(3+1)$-dimensional space satisfies. A $(2+2)$-dimensional space does not satisfy, for the group to which $O(2,2)$ is homomorphic is not simple. These are all the homomorphisms; see Barut and Raczka (1965).

The isomorphism requirement gives a stronger condition. First note that if a function is invariant under the algebra of $O(n)$, it is invariant under that of $CO(n)$, the complex (pseudo-) orthogonal group; the orthogonal group with complex parameters. If we expand a group element (schematically) $R(\theta)=I+\theta J+\cdots$, where J represents the elements of the algebra, then for $CO(n)$, $R(\theta)=I+\theta_r J+i\theta_i J+\cdots$, and we see that the generators are the same for the two groups, since R is orthogonal whether θ is real or complex.

Now a k-component state function in n space is a representation basis state of a unitary-group algebra whose transformations are induced by $CO(n)$ transformations. So the space must allow an isomorphism between the algebras of $CO(n)$ and that of a unitary group that mixes blocks of components (not necessarily all). Another unitary group mixes blocks (of size k/j); it does not commute with $CO(n)$, otherwise it would give a direct product and we would consider the factor group. So that $SU(k)$)

and $CO(n)\times SU(k/j)$ algebras are isomorphic, both giving the full set of continuous transformations of the state function.

What are k, j, and n? The numbers of parameters and of commuting generators must be equal. For the parameters this gives $2\nu(2\nu\pm 1)+(k/j)^2=k^2$, and for commuting generators $\nu+k/j=k$. These must have simultaneous solutions for j, ν, and k, and these all have to be integers. [There are no solutions if the $SU(k/j)$ term is missing.] They have only one integer solution, $k=4, j=2$, and $\nu=2$, with a minus sign. This gives a $(3+1)$-space. Since $O(4)$ is not simple, a 4-space is not possible, for these equations would have to have solutions for two independent 3-spaces, and they do not.

For the Dirac equation, coordinates transform under $CO(n)$, solutions under $U(k)$ and also under the product of $SU(k/2)$ and the unitary group (if any) homomorphic to $CO(n)$. The invariance group is (at most) $SU(k/2)\times CO(n)$. With $n=2\nu$ (or $n=2\nu+1$), $CO(n)$ has $2\nu(2\nu\pm 1)$ parameters. Now the solution is a spinor for which (Boerner, 1963) $k=2^\nu$. Thus the number of parameters of $U(k/2)$ is $2^{(2\nu-2)}$, of $U(k)$ is $2^{2\nu}$, and so of the invariance group of the equation [the sum of the numbers for $CO(n)$ and $U(k/2)$] is $2^{(2\nu-2)}+4\nu^2\pm 2\nu$.

The numbers of parameters of the groups for the equation and the solution are equal (only) for $\nu=2$ and an even-dimensional space, so $n=4$ [but not for $n=3$, so not for $O(3)\times O(3)$, so not for $O(4)$; the group is $O(3,1)$ and space has dimension $3+1$]. Also, the numbers of commuting generators have to be equal, giving $2+k/2=k$. This is satisfied only in $(3+1)$-space. Thus, the Dirac equation is, and can be, form invariant under orthogonal transformations in $(3+1)$-space only.

Interactions (nonlinear terms) act as $U(k)$ transformations on state functions, and so on solutions to the linear equation (say, in-states) they give functions (out-states) that, in the absence of the required isomorphism, are not solutions (and not expandable in terms of solutions—the space given by interactions is larger than that of these solutions). If in-states obey the free-particle equation, there are outgoing one-particle states (with the same spin) that do not obey it. The equation corresponds to a Hamiltonian eigenvalue equation, so the resulting states are not eigenfunctions of the (same) Hamiltonian (as the in-states—and possibly of none). This implies they do not have definite energy (or momentum) and suggests that there is no group for which the state functions are eigenfunctions of physically meaningful representation labels—indicating that the formalism is inconsistent. This implies, without requiring invariance or other transformation properties of the equation, that the nonlinear Diract equation has no solutions in spaces other than $3+1$.

Thus, $(3+1)$-dimensional space is the only one in which the basic requirements of quantum mechanics can be met. What would go wrong if the dimension were other than $3+1$?

If the algebras were not isomorphic, the orthogonal and unitary transformations could not be correlated, so spin or orbital angular momentum components in various coordinate systems could not be consistently related. If the orthogonal had more elements than the unitary algebra, there would be no state-function transformations corresponding to rotations around the x axis, say. Thus, if the angular momentum was along z in one frame, it would be described by the same function in another frame, and so would be along z'. Both frames are equally good; either the description in one frame would be experimentally wrong, or the experimental result would depend on how we wish to describe it. If there were no isomorphism, the result is ridiculous; physics would be inconsistent.

If we arbitrarily related the generators of the two algebras (but there are different numbers of them) and then we wrote a rotation as a product of rotations, different products giving the same rotation would give different components for the spin state function. An experimenter making measurements in two systems gets one result, but making an intermediate measurement, gets a different one (for the angle between spin and velocity, say) with the value depending on what intermediate measurements are made; the final coordinate system is the same, but the values are not.

Consider an observer who sees a particle's spin and momentum (or magnetic field and spin) parallel, queries rotated observers, who see different angles, and finally returns to its own frame and sees different angles, depending on the queries. The observers compare obervations and get conflicting results for the same quantity as seen by the same observer, depending on what comparisons are made. So in a Stern-Gerlach experiment the number and intensities of lines on the screen depend on the angle between the spin and the field (Mirman, 1969). Thus, an observer sees different numbers of lines depending on whether and how he queries rotated observers, although no actions have been performed on the observer, the particles, or any part of the experiment—the queries are all mathematical transformations.

Transformations might be correlated using an isomorphism between the algebras of the orthogonal group and that of a subgroup of a unitary group. Then there are transformations not in the subgroup, whose product is, that give a change of the state function similar to that induced by rotation, so changing the spin direction. A unitary transformation gives a different observer (Mirman, 1979), who relates coordinates by an orthogonal transformation. There are other unitary transformations, so observers, to which there can correspond no orthogonal transformations (the multiplication rules differ). They see different scalar products—there is no consistent way of transforming say, momentum—and, in a Stern-Gerlach experiment, different numbers of lines.

Thus, consistent predictions and laws of physics seem impossible. It is fortunate we live in a space of 3+1 dimensions.

The number of state function components determined by all these conditions (on both the unitary group and its representation and on the Clifford algebra representation) is integral only for dimension 3+1, but for this they all give not only an integer, but the same one. All are satisfied in a space of 3+1 dimensions only.

But for these remarkable coincidences, if any of these numbers were changed by even 1, no universe would be possible. Or would we just have a different formalism?

3. THE PROBLEMS WITH NONRELATIVISTIC QUANTUM MECHANICS

These arguments give the dimension, and signature, of space; quantum mechanics requires relativity. Is nonrelativistic quantum mechanics inconsistent and incomplete?

Consistent theories are based on a few complete, general postulates. Phenomenological theories need *ad hoc* concepts and postulates and rules designed only to get correct answers. These, while useful, are limited and cannot describe (perhaps without further specific rules) phenomena other or more general than those for which they are designed. They do not allow a coherent treatment of the phenomena and consistent incorporation of the concepts and assumptions on which they are based. They are really approximation schemes.

Nonrelativistic quantum mechanics is not a complete, consistent theory. Here we do not consider the requirements for such a theory (since there are no such theories known), but only wish to show that there are inconsistencies, and that *ad hoc* assumptions are needed, and used, for this theory. It is an approximation scheme. To show this, we need not consider how to show when a theory is more than that (so we also need not, and do not, consider whether nonrelativistic quantum mechanics is incomplete in other ways).

The theory is based on the Galilei group (Sudarshan and Mukunda, 1974); why can this not give reasonable physics? The arguments here (and for classical physics) are not as strong as for the dimension where fundamental aspects of well-established theories gave inconsistencies in all but 3+1 dimensions. But the problems here, which one cannot show as rigorously to be unavoidable, are very real and likely inescapable.

Taking (some) of the Galilei group generators as x_i, p_i, and J_i, we define $L_i = \varepsilon_{ijk} \times_j p_k$ and $S_i = J_i - L_i$; S and L commute. Replacing the J's by the L's gives another realization of the group algebra; the S's form an $SU(2)$ algebra. The basis functions of this (expanded) set of generators can be written, suppressing eigenvalues not relevant, as $/l)/s)$, where the first term transforms under the full Galilei group (whose generators are L's, not J's), the second under the $SU(2)$ algebra of the S's. Then $S_i = J_i - L_i$ transforms as a vector under the J's; but this is an extra assumption. The group does not require this expression for S. The group generators can be taken as the L's; the S's are then Galilei-invariant.

So we can rotate coordinates without rotating angular momentum. Thus, two observers measure the momentum, direction of orbital angular momentum, and spins and see different momentum components, but angular momentum and spin the same; at best quite undesirable. Angular momentum is an internal variable in conflict with the definition of orbital angular momentum as resulting from motion in space. This indicates an inconsistency in nonrelativisitic quantum mechanics which is avoided by *ad hoc* postulates. We do rotate angular momentum, because particles are really governed by relativistic equations that have the effect in the nonrelativistic limit of requiring us to choose S to have the commutation relations of a vector with $J \equiv L + S$.

Thus, it is not the group structure, but the extra assumption $S = J - L$ that gives the transformation properties of angular momentum. The mathematics of nonrelativistic quantum mechanics does not require this assumption, so internal orbital angular momentum can be a scalar giving unreasonable physical results.

A particle can have spin (relativity is not needed); but the transformations of it allowed by the mathematics (but ruled out by extra assumptions) give inconsistencies with the definitions and physics. The problem is not the appearance of spin (the difficulties arise with orbital angular momentum and with linear momentum), but what transformations are allowed by the formalism.

The representation describing a particle with spin, or orbital angular momentum, say, a moving H atom, is a direct product of space and spin representations. So space rotations do not induce spin transformations (the free-particle Schrödinger equation does not link spin components; these are internal variables). Also, for two particles the representation is a direct product, so the momentum of one can be rotated without rotating that of the other.

The direct product can be reduced to a sum over irreducible representations. But this does not eliminate the problem. The basis vectors of an irreducible representation are sums of products of two basis vectors each from a different realization (say, orbital angular momentum and spin). Since one can be changed, but not the other (taking a sum of products does not prevent this), the irreducible basis vector does not transform properly, again implying inconsistencies.

For noninteracting particles the Poincaré group representation is a direct product. Does this not give the same difficulty? The classical case (Sudarshan and Mukunda, 1974, p. 466) shows why not. The system's Casimir invariants involve products of variables for different particles. For a rotated observer the invariants, and so the products, are unchanged, so both momenta (and the angular momentum) have to be rotated. Relativistic physics, unlike the nonrelativistic case, couples the quantities. The structure of the Poincaré and Galilei groups differ, leading to the difficulties of a coherent physical theory based on the latter.

This can be seen in other ways. A boost rotates the spin (Wigner rotation; note also the Thomas precession). A rotation obtained from a series of boosts rotates angular momentum. Thus, unlike the Galilei group, coordinate rotations must transform angular momentum. And the Dirac equation, to be invariant, requires that the γ's, so the solution, be transformed; a coordinate transformation rotates spin.

In nonrelativistic quantum mechanics we cannot even say that the state function has relatively complex components; for the free particle there is no linkage between the components, so a particle is in effect a collection of independent scalar particles. At the point the argument for the dimension breaks down (rotations do not induce unitary transformations), so does the theory's physical reasonableness.

So the mathematics on which nonrelativistic quantum mechanics is based, unlike the relativistic case, allows unphysical results. There is no coherent theory that gives only reasonble physics.

Interactions do mix components of the state function. Consider $\sigma \cdot H\psi$, where ψ is a multicomponent object. If the interaction is invariant, a rotation causes a transformation of ψ. However, $SU(2)$ has three parameters, $CO(3)$ has six. There are many $CO(3)$ transformations going with each one of $SU(2)$. A rotation acts on the magnetic field, and also on ψ, and a complex angle gives a change of the phase of the field. A complex rotation has meaning for the magnetic field (which has a phase in an electromagnetic wave). Again there is a product, one factor of which can undergo transformations, while the other cannot.

The state function of a particle in a magnetic field depends on H; its phase depends on that of H and is measurable in an interference (*gedanken?*) experiment, and so the phase of the field is obtained. A rotation through a complex angle mixes components and changes phases and so has experimental consequences. The three field components are functions of six angles, the three complex Euler angles. So the same state function can be found in different ways. A particle can interact with many different fields and have the same final state. It is impossible for the final state to contain complete information about the interaction.

Thus, nonrelativistic quantum mechanics is not able to describe a system properly, without adjustments; it cannot give a complete theory of these interactions, nor of matter.

It may be surprising that this is not generally known. However, one usually does not consider complex rotations (under which Schrödinger's equation is invariant). Also, since systems obey relativistic equations, there is only one correct solution (with no ambiguity). This is the one chosen without it being explicit that there is a choice.

The interaction of a spinning particle with a magnetic field is described quite well in this framework (we know how to avoid the problems). But a complete theory containing everything about any system is not possible; all the information cannot be put in. (We do not consider possible mathematical contradictions.) Nonrelativistic quantum mechanics does not have enough structure to be a complete, coherent, consistent framework [as shown by the direct-product description of multiparticle states and the lack of a large enough unitary group to give a homomorphism with $CO(n)$]. It gives consistent results in restricted domains; it is an approximation to a more complete theory—relativistic quantum mechanics, which its foundations require. This is not a complete, coherent, consistent theory either. But it is closer.

4. WHY CLASSICAL PHYSICS IS NOT CORRECT

If nonrelativistic quantum mechanics is purely phenomenological, can classical physics be anything more? Classical physics is not completely defined. There is always the (quite unlikely) possibility that with the proper definition of terms a complete description of a simple system might be found that could be called classical. It would not be experimentally correct and there are good reasons why. We indicate some; these make implausible a reasonable classical physics.

In a certain sense classical physics (and nonrelativistic quantum mechanics) has been extremely successful. Does this contradict the view that it is impossible (as a consistent theory; of course it is quite possible, up to a point, as a phenomenological one)?

Classical physics does not describe the universe beyond a certain level; the question is whether it could. There is only one classical interaction, electromagnetism (which requires relativity). But this is inconsistent, having problems, for example, with infinite energy and radiative reaction; the Rayleigh–Jeans law for a black-body gives infinite energy. Thus, there is no known consistent classical theory.

How about simple systems? A collection of point particles interact only when they collide. But point particles never collide, and so never interact.

A consistent theory of this may be possible, but it is so trivial as to be pointless. Also, in flat space there would be infinite dispersal (containing walls made of point particles are impossible). Curved space would here be an *ad hoc* assumption; the theory would be at best contrived—and trivial.

Hard spheres (again having infinite dispersal) could be used, presumably with no internal structure and infinite hardness; only their momentum would change during collisions, not shape or size. A system of infinitely hard, finite-size objects would raise questions (including those about assumptions of coefficients of restitution, and sizes and shapes being *ad hoc*). There are no bound states, so even if we regarded the system as possible and consistent, it would give a trivial universe.

Another possibility is point particles with attractive forces (repulsive forces would cause them to move infinitely far apart in a flat space; curvature would again be contrived). Anything more than $1/r^k$ would introduce arbitrary distances (even k is arbitrary). There are problems with instantaneous action at a distance unless we introduce fields, but then we get into questions such as radiation reaction, and so on. Also, there are problems with infinite energy. There is a way of dealing with this—ignore it. This is reasonable for phenomenological theories, but not for complete and consistent ones.

For Coulomb potentials matter would collapse (see Lieb, 1976, for review), except for the Pauli exclusion principle, which is not relevant to a classical theory. The analysis is quantum mechanical (and three-dimensional), leaving a slight possibility that there could be stable classical matter if parameters were chosen correctly, but there is no reason to think so. A universe that collapsed to a point would not be interesting.

Thus, except perhaps for trivial, uninteresting, artificial cases, classical physics is unlikely to provide a framework for a reasonable theory of matter; even simple models run into difficulties. It is useful as pure phenomenology in certain domains, but cannot be pushed beyond that. It is not an alternate framework to quantum mechanics that happens to disagree with experiment, and is not incomplete, because there is no theory of matter to use with it. Rather, classical physics is an inadequate and so impossible setting for theory of any realistic universe; it does not have a rich enough structure for that.

What is wrong? Classical physics suffers the same problems as nonrelativistic quantum mechanics (where relevant), but there are worse ones. Consider the functions describing a free particle, velocities and the Hamiltonian, say. These, by translation invariance, are required to be eigenfunctions of the momentum operators, $p \sim d/dx$. They are, but the eigenvalues are all zero; they are all constant. Thus, momentum operators (with similar difficulties for angular momentum) are realized trivially (so the representations of the Galilei group are limited). This is insufficient.

The states, the group representation basis vectors, need labels that describe the system, such as momentum and angular momentum. This is not sufficiently possible in classical physics. To express properly the transformation properties required by the nature of space, systems should be completely labeled by eigenvalues of the group over space. But there is no way of distinguishing in this manner a particle with one momentum from that with another. Thus, classical physics cannot even properly describe the simplest case, a single free particle.

Why does this matter? Interactions change the state of a system; acting on one basis vector, they give another. However, this cannot be done here, for there is no way properly to describe the state. There is no basis vector on which an interaction can act (the objects of classical physics are numbers, not basis vectors). Thus, interactions can be included in the framework only phenomenologically.

As the example of the momentum operators shows, classical physics is not rich enough to describe nature fully. One reason it cannot encompass a complete theory is that it cannot properly express a fundamental property of space, how it is transformed.

So quantum mechanics is not arbitrary. It seems (and probably is) necessary.

Neither nonrelativistic quantum mechanics nor classical physics provides counterexamples to the present considerations.

5. GROUP THEORY AND THE FOUNDATIONS OF QUANTUM MECHANICS

Group theory and consistency imply that neither classical physics nor nonrelativistic quantum mechanics is coherent and completeable. Can they say more? To understand how much of the formalism comes from these requires detailed analyses and studies of experiment and measurement, which is not possible here. We only indicate how this type of analysis might provide deeper insight.

The way to construct a more complete theory is clear (especially since we know the answer). Objects transform under the inhomogeneous group of the space. The operators of this group are realized (most simply, although if we did not know the answer, might we use other realizations, if possible?) as first-order differential operators (the objects are functions of space). So momentum $p \sim i\,d/dx$, with the i to give bounded eigenfunctions.

From translation invariance a free-particle state function (a basis vector of an inhomogeneous-group representation, with diagonal inhomogeneous operators, p's) is expandable in terms of translation eigenfunctions. A state-function (or operator, the picture is irrelevant) at a point is given by the one at the origin acted on by translations.

That p^2 is an invariant (called mass—giving the Klein-Gordon equation) gives the relationship between momentum and velocity. In the rest system $E = m$ (c, like h, is just a conversion factor). Momentum and energy in other systems are given by Lorentz transformations; this gives their dependence on velocity (the expressions for momentum and energy, the generator eigenvalues, as functions of the transformation parameter v are completely group-theoretic). The free-particle wave function gives the relationship between momentum and wavelength, and energy and frequency.

Thus, the fundamental kinematical variables must be related as they are because of the transformation properties of space. These are not assumptions; they come from the requirements on the formalism.

Why does the state function give a probability? Consider a free particle described by a wave packet—a function of space with various momenta, thus velocities (from this discussion and Fourier analysis). We repeatedly prepare a particle with the same velocity and position distribution and then measure the position at a different time. If all second positions were the same, the velocities and positions would be known exactly for the first measurement. But the state function depends on different velocities—the second positions must vary. The state function would then (can there be another choice?) give the number of times each position was found, its probability. The crucial point is that consistency requires different results for the value of the same quantity measured in a series of identical experiments.

The probability depends on the absolute value of the state function. Why? It has to be determined through many measurements, so the phase averages to zero. Why the square? An operator on a function gives a number times a function. An expectation value (a representation matrix element; so the function appears twice) is a number, not a function. To eliminate the function, we take its product with another function, for diagonal elements with itself, giving the square.

The expectation value is the number of states with the same eigenvalue times the value, summed over all values (and normalized). So the number of states having a particular eigenvalue—the probability—depends on the state function squared. Thus, the probability is proportional to the state function squared and, since the phase averages to zero, the absolute square of the state function.

This does not rigorously show that a state function has to be interpreted as giving a probability, but there seems no alternate interpretation, so a consistent formalism might actually require this.

Is spin required? Not for the counting arguments for the dimension, since they hold for nonzero orbital angular momentum (which would be difficult to avoid). But there cannot be only scalar particles (taking all coordinates symmetrical).

There are relations between momentum eigenvalues since a function of them is invariant. The Hamiltonian, whose eigenequation gives the wave equation, must be a function of the others. It is proportional to the first-order derivative in the time and by symmetry all coordinates enter in the same way, so the Hamiltonian depends on first-order derivatives with respect to each coordinate. The wave equation is invariant, and so is $\Gamma_i\, d/dx_i$, where the Γ's are operators (they cannot be pure numbers, for then p^2, calculated from these p's, would not be an invariant), and so can be represented by matrices. The Γ's are shuffled by rotations if the equation is invariant; the components of the Γ's, which must have several components, must be mixed, and so, therefore, must those of the state functions on which they act. There are no operators that give only scalars, since they would have only a single component. Nonzero spins must appear if there are any particles at all.

Thus much (all?) of the formalism of quantum mechanics follows from consistency, invariance, and assumptions (difficult to avoid) about the realization of the group operators; it is not arbitrary. This does not (yet) give a complete theory (no complete theory is known), but does lead to the dimension, relativity, and places strong restraints on how we describe nature.

So quantum mechanics, which seems to pose serious puzzles and paradoxes, may really be a simple, comprehensible theory (Mirman, 1973a,b, 1985), completely natural.

If two particle (of equal mass in the center-of-mass system, for simplicity) collide, conservation of the four components of momentum for masses of the particles unchanged requires that the magnitude of the three-momentum be unchanged. Thus, the only result of the interaction would be a change in direction. So, anything beyond the most trivial physics in relativistic quantum mechanics leads to particle creation and annihilation, thus quantum field theory.

A Hamiltonian (time-translation operator) eigenfunction that contains a two-particle state must be a sum of states with different numbers of particles. This implies the machinery of field theory. Of course, quantum field theory has its own consistency problems. It is not a complete, consistent, coherent theory either. At present all theories are phenomenological, some more than others.

While these consistency arguments help us understand why the universe is described by quantum mechanics, and why it has the form and meaning it does, we should be careful about pushing this too far. Otherwise, we will show that the current theory is the only one possible, at which point experiment will show that it is wrong. However, these arguments help clarify how the various parts of the theory are related, and they indicate underlying reasons for the nature of the universe.

6. WHAT MUST WE ASSUME ABOUT PHYSICS!

Relativity, quantum mechanics, field theory, the dimension and signature of space, and probably much else are implied or required by properties of space, matter, and physical laws so basic that it seems unthinkable for them to be different. It may be (although it is probably unprovable) that a consistent universe is unique.

These results are founded on assumptions about nature and how we understand it. What are they? Undoubtedly, there are many we do not realize. If we cannot recognize all our assumptions, as of course we cannot, it is still useful to state those we can, no matter how vague they may be, or how vaguely they must be presented; there are many fundamental problems in this area that need study, and perhaps listing some here will at least stimulate further thought.

I assume all standard mathematical principles and formalisms (an assumption about what mathematical formalism correctly describes the physical universe) whatever this may involve and attempt to list the physical assumptions. Not all are needed to obtain each result.

A. First is the existence of observers. Living, intelligent observers are essential for physics to have meaning. However, while these assumptions may be essential for intelligence, even for life, we wish to find those essential for physics—for a set of consistent laws governing the universe. Observers are objects that have properties with respect to which the properties of other objects can be measured such that the relationship between the properties can be defined in a consistent way. Spin or momentum of a particle determines a direction, for example. This, especially the definition of frames using quantum mechanical particles, has been discussed previously (Mirman, 1975).

B. Mathematically, coordinate systems can be defined in a space. We assume these can be defined physically and that it is possible to have observers in many of them—there is no physical reason why they cannot exist. (The question of whether this last sentence makes one statement or two is left open.) That is, we assume physical objects with different values of a quantity (say, particles with different momenta or spin directions). Each set of values defines a frame of reference (thus the momenta of a set of particles determine a system). We assume that there are transformed observers, physical objects—whatever these may be—in different systems (that is their existence would not lead to contradictions). Observers might exist in only a subset of systems—we can define mathematically the values of a quantity that no physical object can have; superluminal frames are mathematically definable, but cannot contain physical objects.

We do not require that the laws of physics be the same in all systems, only that it be possible to make, and compare, measurements with respect to different ones and that the results in one system be a function of those in another. The function, assumed determinable, depends on the relationship between the systems (the angles, say). We require that these relationships not lead to inconsistencies.

Also we assume (without analyzing the physics involved) that the transformations between coordinate systems (observers) form a group, a Lie group (presumably with transitive action).

We expect there to be quantities transforming as the coordinates (velocity and momentum, say) and others (like spin) affected by a state-function transformation. We require that (there are) scalar products of these (that) are invariant—they transform as the identity representation, an invariant of the group over space. The requirement is not that products have the same values in all systems, but that they be the same functions of the variables. Without such products peculiar, if not impossible, physics, and also mathematical inconsistencies, are likely.

An alternate requirement is that the equations of motion (the Dirac equation, for example) be form-invariant under transformations.

If scalar product is not invariant in a transformed system, it does not go into the same function with a different value, but into a different function; observations in different systems would be markedly different and likely inconsistent. This invariance requirement may have hidden assumptions. Invariance is ascertained by measurement requiring physical laws that prescribe the measurement process. If these were to change too radically, presumably invariance would become meaningless. Even such requirements as the number of particles looking the same for transformed observers brings in (hidden) assumptions about laws and measurement.

C. If not every system need have observers, every group generator must connect systems in which they are possible. Otherwise we could rotate around some axes, but not all; we would not be able to rotate in the xy plane without (at least) also rotating in the zt plane. Rotations would be impossible without also causing an object to move. There would be no observers at rest with axes different from some fixed one. The spins of electrons at rest would all be in the same direction. There would be no physical meaning of, or consistent way to measure with respect to, other axes. If observers in all systems were not possible, either there could be no objects whose spin would point in any direction except for a fixed one or, if there were such objects, angles (and scalar products) could not be measured with respect to them. Space would not be isotropic. Consistency would be doubtful.

D. Space is defined using the transformation group (perhaps this is more fundamental); a basis vector of a (linear) representation is defined at one point, and the vector at another point (which is close enough to the first so that we can regard space as flat) is given by a translation operator. So the vector is a function of parameters, taken as the coordinates. Thus, it is fundamental that it be possible to relate the states at different points by means of group operators.

E. There are representations with bases on which the generators give eigenvalues interpretable as describing particles. There are also multiparticle states; product representations whose eigenvalues can be interpreted as quantities describing a set of particles and have eigenvalues of operators that can be interpeted as describing each of the particles. What particles are or whether they are necessary is not considered. But to describe the physical reasons for mathematical procedures and assumptions particle terminology is used.

The homogeneous part of the group algebra is taken as decomposable into a product of realizations, each describing a single particle; the generators are sums, each term acting on a single particle (changing a relationship, e.g., distance or angle, between particles).

The argument above that the state function gives a probability assumed that values at one point and time are related to those at another by means of state-function transformations, which is natural when considering particles and their coordinates and momenta. How serious this assumption is remains unclear—that is, can there be consistent physics in which there are no such things as particles, and in which the experiments considered were not possible? Continuous fluids are often reasonable approximations. But is it possible to have a complete theory based only on them, one in which every possible question can be answered using only a small number of properties?

F. The particles are assumed to interact (it would be a dull world without interactions). That is, the translations are nonlinear operators; they are nonlinear functions of basis vectors (nonlinear in that acting on one-particle states they produce sums of these plus multiparticle states; in the language of potentials, the state function, a function of space, is multiplied by another function of space, the potential). In particular, the time-translation operator acting on a state gives a linear combination of the same plus other states.

This whole analysis rest heavily on the existence and properties of transformations, raising the question of why they are so central. Consider an object defining a coordinate system (Mirman, 1979); the momentum is determined by the system the particle is in, but the momentum can also be taken to be one of the values defining the system (Mirman, 1979). The object undergoes interactions that change its properties, therefore changing the coordinate system it defines; we can measure with respect to the particle before or after an interaction. If there were not a set of allowable systems, it would be impossible for an object to move from one to another, implying that it could not undergo interactions, or that any interaction would so change the object that while it was possible to measure with respect to it before, it was not possible afterward, which is likely to be meaningless.

Thus, the existence and properties of transformations are implied by the existence of interactions. This cannot be explained now, but it is reasonable that this analysis can be redone without transformations to show that a set of equations describing interacting objects, rich enough to give a reasonable physics, can be consistent only under the conditions found here.

The Dirac equation with interactions, while not analyzed from this point of view, likely can only be consistent in $3+1$ dimensions.

This implies that a coherent dynamical theory probably cannot be independent of a theory of matter. The conditions (many may not yet be known) may be such that we cannot write down an equation of motion and put in arbitrary interaction terms; the equations and interactions are unique.

So the essential point is not transformations, but consistency. Transformations are a way of probing its consequences, but perhaps not the only way. Or perhaps transformations and symmetry represent the essence of interactions and their effects and so may have special significance. Again this is worthy of more study.

7. DISCUSSION

What, and how strong, are the constraints on possible physical laws? Why do they exist? Are they simply the result of the way we look at the world, of our own patterns of thought and neural organization? Or might there be actual restrictions (how limiting?) on nature, on possible laws and universes?

Ultimately these questions are unanswerable. Yet it is possible to understand the constraints on our formalism (self-imposed?) and on the universe it can describe and to derive many properties of nature. It is also possible that our formalism has (or must have) within it such strong restrictions that it can describe only a single universe.

The laws of physics are consistent and coherent, and simple. This is hardly surprising, but it is strongly emphasized by these results. The dimension and signature of space are determined by very basic principles through several, closely related, conditions. If any of these was not satisfied, it is doubtful that there could be consistent laws of physics, thus a consistent universe. Yet they all hold, and all give the same result.

The universe is highly improbable. But it exists.

This suggests some interesting speculation. There is a view that the reason the universe is hospitable to life and intelligence is that otherwise we would not be here to know how hospitable it is. And of course in a way this is true. In an oscillating universe in which conditions and perhaps laws change in each cycle, it is reasonable that occasionally there will be a cycle in which intelligence exists. And the intelligence will then discover that the laws of nature give a universe hospitable to intelligent life.

However, the present considerations suggest that the laws are severely limited, they can change at most slightly from cycle to cycle; that there is even a possible dimension that allows the universe to exist seems accidental. Unless the laws of arithmetic can change also between cycles, this approach seems unable to provide an explanation for there being any universe at all.

Or, could it simply be that many universes are possible and that our formalism is designed to describe the one in which we live? It can describe no other, but that is its fault, not that of the universe.

Transformations and coherence lead to quantum mechanics, relativistic quantum mechanics, and field theory. But none of these is rich enough to allow a complete theory of matter. Perhaps the search for such a theory should be guided by one general, quite useful principle, consistency.

ACKNOWLEDGEMENT

Conversations with Prof. Abner Shimony stimulated thinking on many of these topics.

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